J. Korean Math. Soc. **45** (2008), No. 3, pp. 835–840

UNIMODULAR ROOTS OF RECIPROCAL LITTLEWOOD POLYNOMIALS

Paulius Drungilas

Reprinted from the
Journal of the Korean Mathematical Society
Vol. 45, No. 3, May 2008

J. Korean Math. Soc. **45** (2008), No. 3, pp. 835–840

UNIMODULAR ROOTS OF RECIPROCAL LITTLEWOOD POLYNOMIALS

Paulius Drungilas

Abstract. The main result of this paper shows that every reciprocal Littlewood polynomial, one with $\{-1, 1\}$ coefficients, of odd degree at least 7 has at least five unimodular roots, and every reciprocal Littlewood polynomial of even degree at least 14 has at least four unimodular roots, thus improving the result of Mukunda. We also give a sketch of alternative proof of the well-known theorem characterizing Pisot numbers whose minimal polynomials are in

$$\mathcal{A}_N = \left\{ X^d + \sum_{k=0}^{d-1} a_k X^k \in \mathbb{Z}[X] \,:\, a_k = \pm N,\, 0 \leqslant k \leqslant d-1 \right\}$$

for positive integer $N \geqslant 2$.

1. Introduction

In this paper we study unimodular roots of reciprocal Littlewood polynomials (those with coefficients in $\{-1, 1\}$). Borwein, Erdélyi, and Littmann [2] proved that any polynomial in

$$K_n := \left\{ \sum_{k=0}^{n} a_k X^k \,:\, |a_0| = |a_n| = 1 \text{ and } |a_k| \leqslant 1 \right\}$$

has at least $8\sqrt{n}\log n$ zeros in ball with center on the unit circle and radius $33\pi\frac{\log n}{\sqrt{n}}$. Thus polynomials from K_n have quite many zeros near the unit circle. One may naturally ask how many unimodular roots a polynomial in K_n can have? Clearly, every Littlewood polynomial is in K_n. Mercer [13] proved that some special Littlewood polynomials, the so called skewsymetric polynomials, have no roots on the unit circle. Thus some additional constraints must be layed on a polynomial from K_n for it to have unimodular roots. Usually one requires a polynomial to be reciprocal (a polynomial $f(X)$ of degree d is called reciprocal if $f(X) = X^d f(1/X)$). Lakatos and Losonczy [11] considered reciprocal polynomials having all their zeros on the unit circle. More precisely, they proved

Received October 19, 2006; Revised December 20, 2006.

2000 *Mathematics Subject Classification.* 11R06, 11C08, 30C15, 12D10.

Key words and phrases. Pisot numbers, Littlewood polynomials, unimodular roots, reciprocal polynomials.

that all zeros of the reciprocal polynomial $P_m(X) = \sum_{k=0}^{m} a_k X^k \in \mathbb{C}[X]$ of degree $m \geqslant 1$ are on the unit circle if $|a_m| \geqslant \frac{1}{2}\sum_{k=1}^{m-1} |a_k|$ (see also [8], [9], [10]). Recently Schinzel [16] generalized this result. Konvalina and Matache [6] obtained some sufficient conditions for a reciprocal polynomial to have at least one unimodular root.

Borwein, Erdélyi, and Littmann [2] asked what is the minimum number of unimodular roots of reciprocal Littlewood polynomial of degree n? Recently Borwein, Erdélyi, Ferguson, and Lockhart [1] proved that the average number of unimodular roots of reciprocal Littlewood polynomials of degree n is at least $n/4$. It is believed that the minimum number of unimodular roots of such polynomials tend to infinity with n, but this does not appear to be easy. Erdélyi [5] proved that every reciprocal Littlewood polynomial has at least one unimodular root (see also [13]). Recently Mukunda [14] improved this result for odd degree reciprocal Littlewood polynomials by showing that every reciprocal Littlewood polynomial of odd degree at least 3 has at least three unimodular roots. The main result of this paper is

Theorem 1. *Every reciprocal Littlewood polynomial of odd degree $d \geqslant 7$ has at least five unimodular roots. Every reciprocal Littlewood polynomial of even degree $d \geqslant 14$ has at least four unimodular roots.*

Theorem 1 sharpens the result of Mukunda. The bounds on d of this theorem are sharp, since the polynomial $q_2(X) = X^5 - X^4 - X^3 - X^2 - X + 1$ has exactly three unimodular roots, and the polynomial $p_6(X) = X^{12} + X^{11} + X^{10} + X^9 + X^8 + X^7 - X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ has exactly two unimodular roots. The key tools in the proof of Theorem 1 is Lemma 5 and Theorem 2.

A *Pisot number* is a real algebraic integer $\alpha > 1$, all of whose conjugates lie inside the open unit disc. The set of all Pisot numbers is usually denoted by $\mathcal{S}$. This set is known to be closed (see [15]), and its minimum is known to be the largest root of $X^3 - X - 1$, which is approximately $1.324717\cdots$ (see [17], [18]). Mukunda [14] determined all Pisot numbers whose minimal polynomials are Littlewood polynomials. The smallest such Pisot number is the golden ratio $(1+\sqrt{5})/2$ (see [3]). Denote

$$\mathcal{A}_N = \left\{ X^d + \sum_{k=0}^{d-1} a_k X^k \in \mathbb{Z}[X] \,:\, a_k = \pm N,\, 0 \leqslant k \leqslant d-1 \right\}$$

for positive integer $N \geqslant 2$. Next theorem is well-known ([19] p.51–52, [7]) It determines all Pisot numbers whose minimal polynomials lie in $\mathcal{A}_N$.

Theorem 2. *Let γ_n be a Pisot number of degree n whose minimal polynomial $P_n(X) \in \mathcal{A}_N$. Then*

$$P_n(X) = X^n - NX^{n-1} - NX^{n-2} - \cdots - NX - N.$$

The sequence γ_n is strictly increasing and converges to $N+1$.

Mukunda [14] proved this theorem for $N = 1$ using an algorithm constructed by Boyd [4] for determining all Pisot numbers in a real interval. In Section 2, we give a sketch of alternative proof of Theorem 2 which is straight generalization of Mukunda's proof.

2. Pisot numbers: the proof of Theorem 2

Our proof of Theorem 2 is straight generalization of proof of Theorem 1 in [14] (Section 3). Thus we describe only the deviations of this proof.

The proof of the next lemma can be found in [12].

Lemma 3. *Let $f(X) = \sum_{k=0}^{n} a_k X^k$ be a polynomial with complex coefficients, with $a_n \neq 0$. If $f(\zeta) = 0$, then $|\zeta| < 1 + \max\{|a_k/a_n| : 0 \leqslant k \leqslant n-1\}$.*

Proof of Theorem 2. Let α be a Pisot number whose minimal polynomial $P(X) \in \mathcal{A}_N$. Then $\alpha \geqslant N$, since $P(0) = \pm N$. Lemma 3 implies that $\alpha \leqslant N+1$. Thus, all Pisot numbers, whose minimal polynomials are in $\mathcal{A}_N$, must lie in the interval $[N, N+1]$.

Now we use an algorithm described in [14] (Section 2) to construct Pisot numbers from interval $[N, N+1]$ whose minimal polynomials are in $\mathcal{A}_N$. We claim that

1. $D_k(X) = -X^k + NX^{k-1} + \cdots + NX + N$ for $k \geqslant 1$,
2. $w_k = N(N+1)^k - N - 1$ for $k \geqslant 1$,
3. $w_k^+ \geqslant N(N+1)^k + N + 1$ for $k \geqslant 2$,
4. $v_k^+ = N(N+1)^k + 1$.

The proof of these claims is by induction on k. First, we work out the first two cases ($k = 1, 2$).

Case $k = 1$. Polynomial $D_1(X) = -X + d_1 \in \mathcal{A}_N$ gives Pisot number if and only if $d_1 = N$. Thus, $u_0 = N$, $E_1(X) = -NX + 1$ and $w_1 = d_1^2 - 1 = N^2 - 1$. Similarly, $D_1^+(X) = X + N$ and $E_1^+(X) = NX + 1$.

Case $k = 2$. Let $D_2(X) = -X^2 + d_1X + d_2$ and $E_2(X) = -d_2X^2 - d_1X + 1$. Then we have $D_2(X)/E_2(X) = N + u_1X + w_2X^2 + \cdots$. Thus $D_2(X) = E_2(X) \cdot \left(N + u_1X + w_2X^2 + \cdots\right)$. Comparing coefficients we obtain $d_2 = 2$, $u_1 = (N+1)d_1$ and $w_2 = u_1^2/(N+1) + N^2 - 1$. We wish $D_2(X)$ to lie in $\mathcal{A}_N$. Thus $d_1 = \pm N$, $u_1 = \pm N(N+1)$ and inequality $u_1 \geqslant u_0^2 - 1 = N^2 - 1$ implies $u_1 = N(N+1)$. Now we have $D_2(X) = -X^2 + NX + N$, $E_2(X) = -NX^2 - NX + 1$ and $w_2 = N(N+1)^2 - N - 1$. It is easy to check that $D_2^+(X) = X^2 - \frac{N(N+1)}{N-1}X + N$, $E_2^+(X) = NX^2 - \frac{N(N+1)}{N-1}X + 1$, $w_2^+ = N^2(N+1)^2/(N-1) - N^2 + 1 > N(N+1)^2 + N + 1$. Finally, we obtain $v_2^+ = N(N+1)^2 + 1$ (see equality (10) in [14]).

We skip the induction step since it is analogous to that of Mukunda's. As in [14] we use $w_k \leqslant u_k \leqslant v_k^+$ to obtain inequalities for u_k : $N(N+1)^k - N - 1 \leqslant u_k \leqslant N(N+1)^k + 1$. To obtain $u_k = N(N+1)^k$, we need

Lemma 4. *If $(u_0, u_1, \dots, u_n)$ is a terminal node giving Pisot number whose minimal polynomial $P(X) \in \mathcal{A}_N$, then $u_j \equiv \pm N \pmod{N^2}$ for $0 \leqslant j \leqslant n-1$.*

Proof of Lemma 4. We have

$$D_n(X) = -P(X) = -X^n + d_1 X^{n-1} + \cdots + d_{n-1} X + d_n,$$

$$E_n(X) = -d_n X^n - d_{n-1} X^{n-1} - \cdots - d_1 X + 1,$$

$$\frac{D_n(X)}{E_n(X)} = u_0 + u_1 X + \cdots + u_{n-1} X^{n-1} + \cdots . \tag{1}$$

The coefficients $d_k = \pm N$, $1 \leqslant k \leqslant n$, since $P(X) \in \mathcal{A}_N$. Multiplying (1) on both sides by $E_n(X)$ and comparing the coefficients of X^k for $0 \leqslant k \leqslant n-1$, we obtain

$$d_{n-k} = u_k - u_{k-1} d_1 - \cdots - u_0 d_k . \tag{2}$$

Finally, for $0 \leqslant k \leqslant n-1$, equalities (2) modulo N^2 imply $u_j \equiv \pm N \pmod{N^2}$. □

Thus for each positive integer n we have exactly one Pisot number γ_n of degree n whose minimal polynomial is in $\mathcal{A}_N$. It is a root of $D_n(X) = N + NX + \cdots + NX^{n-1} - X^n$. Further, it is easy to check that $\frac{n}{n+1}(N+1) < \gamma_n < N+1$. Thus $\gamma_n \to N+1$, $n \to \infty$. Finally, noting that for positive integer $n \geqslant 2$ $D_n(X)$ is positive in the interval $[1, \gamma_n)$, negative in the interval (γ_n, ∞) and $D_n(\gamma_{n-1}) = \gamma_{n-1}^{n-1}(N + 1 - \gamma_{n-1}) > 0$, we obtain $\gamma_{n-1} < \gamma_n$. This completes the proof. □

3. Unimodular roots of reciprocal Littlewood polynomials

The proof of the following lemma can be found in [14].

Lemma 5. *Suppose $p(X)$ is a polynomial in $\mathbb{C}[X]$, m is a positive integer and w is a complex number of modulus one. Then the number of roots of $R_m(X) = wX^m p(X) \pm p^*(X)$ in the closed unit disk is greater than or equal to the number of roots of $S_m(X) = X^m p(X)$ in the same region.*

Denote $p_n(X) = X^{2n} + X^{2n-1} + \cdots + X^{n+1} - X^n + X^{n-1} + \cdots + X + 1$ and $q_n(X) = X^{2n+1} - X^{2n} - \cdots - X + 1$. The next lemma will be used in the proof of Theorem 1.

Lemma 6. *Polynomial $q_n(X)$ has at least $2n-1$ unimodular roots and polynomial $p_n(X)$ has at least $(2n-8)/3$ unimodular roots.*

Proof of Lemma 6. Denote $r_n(X) := (X-1)\cdot q_n(X) = X^{2n+2} - 2X^{2n+1} + 2X - 1$ and $s_n(X) := (X-1)\cdot p_n(X) = X^{2n+1} - 2X^{n+1} + 2X^n - 1$. Substituting $X = e^{it}$

into these polynomials we obtain

$$\begin{aligned} r_n(e^{it}) &= (-2\sin(n+1)t + 2i\cos(n+1)t)\,(\sin(n+1)t - 2\sin nt) \\ s_n(e^{it}) &= \left(-2\sin\frac{2n+1}{2}t + 2i\cos\frac{2n+1}{2}t\right)\times \\ &\quad \times\left(\sin\frac{2n+1}{2}t - 2\sin\frac{t}{2}\right). \end{aligned}$$

Now counting the number of changes of sign of the function $f(t) = \sin(n+1)t - 2\sin nt$ we see that function $r_n(e^{it})$ has at least $2n$ zeros in the interval $[0,\, 2\pi)$. Indeed, function $f(t)$ changes its sign in the interval

$$\left[\frac{\pi k + \pi/2}{n},\ \frac{\pi(k+1)+\pi/2}{n}\right]$$

for $k = 0, 1, \ldots, 2n-1$. Thus polynomial $q_n(X)$ has at least $2n-1$ unimodular roots.

We can now proceed analogously to the proof of the second part of lemma. Denote $\phi(t) = \sin\frac{2n+1}{2}t - 2\sin\frac{t}{2}$. The number of zeros of $\phi(t)$ in the interval $[0,\, \pi/3]$ equals the number of zeros in the interval $[5\pi/3,\, 2\pi]$, since $\phi(2\pi - t) = \phi(t)$. Now counting the number of changes of sign of the function $\phi(t)$ in the interval $(0,\, \pi/3]$ we see that $\phi(t)$ has at least $(n-4)/3$ zeros in this interval. Indeed, function $\phi(t)$ changes its sign in the interval $\left[\frac{2\pi k+\pi}{2n+1},\, \frac{2\pi(k+1)+\pi}{2n+1}\right] \subset (0,\, \pi/3]$ for $k = 0, 1, \ldots, \left[\frac{n-4}{3}\right]$. Then the function $\phi(t)$ has at least $(2n-8)/3$ zeros in the interval $(0,\, 2\pi)$. Thus the polynomial $p_n(X)$ has at least $(2n-8)/3$ unimodular roots. □

Proof of Theorem 1. Mukunda [14] proved that reciprocal Littlewood polynomials of odd degree at least five, except possibly those of the form $q_n(X) = X^{2n+1} - X^{2n} - \cdots - X + 1$, must have at least five unimodular roots. Now Lemma 6 for $n \geqslant 3$ implies that polynomial $q_n(X)$ has at least five unimodular roots.

Suppose that $F(X) = \sum_{j=0}^{2n} a_j X^j$ is a reciprocal Littlewood polynomial of even degree. Erdélyi [5] proved that every reciprocal Littlewood polynomial has at least one unimodular root. Then $F(\pm 1) \neq 0$, since $F(\pm 1)$ is an odd integer. Thus the number of unimodular roots of $F(X)$ is even. Assume that $F(X)$ has exactly two unimodular roots. Then $F(X)$ has $n-1$ roots inside the open unit disc. Denote $p(X) = X^n + \sum_{j=0}^{n-1} 2a_j a_n X^j$. Then $p(X) \in \mathcal{A}_2$ and $F(X) = a_n/2\,(X^n p^*(X) + p(X))$. Applying Lemma 5 we obtain that $p^*(X)$ must have exactly one root in the closed unit disc. Thus $p(X)$ must have exactly one root ζ outside the open unit disc and this root must be real. Then Theorem 2, for $N = 2$, implies $p(X) = D_n(X)$ if $\zeta > 1$, and $p(X) = (-1)^n D_n(-X)$ if $\zeta < -1$. Here $D_n(X) = X^n - 2X^{n-1} - \cdots - 2X - 2$. Thus reciprocal Littlewood polynomials of even degree at least four, except possibly those of the form $p_n(X) = X^{2n} + X^{2n-1} + \cdots + X^{n+1} - X^n + X^{n-1} + \cdots + X + 1$, must have

at least four unimodular roots. Applying Lemma 6 for $n \geqslant 10$ we obtain that $p_n(X)$ has at least $(2n-8)/3 \geqslant 4$ unimodular roots. Finally, a simple computation with Maple shows that for $n = 7, 8, 9$ polynomial $p_n(X)$ also has at least four unimodular roots. $\square$

Acknowledgements. This research was partially supported by the Lithuanian State Science and Studies Foundation.

References

[1] P. Borwein, T. Erdélyi, R. Ferguson, and R. Lockhart, *On the zeros of cosine polynomials: solution of an old problem of Littlewood* (submitted).

[2] P. Borwein, T. Erdélyi, and F. Littmann, *Zeros of polynomials with finitely many different coefficients*, Trans. Amer. Math. Soc. (to appear).

[3] P. Borwein, K. G. Hare, and M. J. Mossinghoff, *The Mahler measure of polynomials with odd coefficients*, Bull. Lond. Math. Soc. **36** (2004), no. 3, 332–338.

[4] D. W. Boyd, *Pisot and Salem numbers in intervals of the real line*, Math. Comp. **32** (1978), no. 144, 1244–1260.

[5] T. Erdélyi, *On the zeros of polynomials with Littlewood-type coefficient constraints*, Michigan Math. J. **49** (2001), no. 1, 97–111.

[6] J. Konvalina and V. Matache, *Palindrome-polynomials with roots on the unit circle*, C. R. Math. Acad. Sci. Soc. R. Can. **26** (2004), no. 2, 39–44.

[7] P. Lakatos, *On a number theoretical application of Coxeter transformations*, Riv. Mat. Univ. Parma (6) **3** (2000), 293–301.

[8] ———, *On polynomials having zeros on the unit circle*, C. R. Math. Acad. Sci. Soc. R. Can. **24** (2002), no. 2, 91–96.

[9] ———, *On zeros of reciprocal polynomials*, Publ. Math. Debrecen **61** (2002), no. 3-4, 645–661.

[10] P. Lakatos and L. Losonczy, *On zeros of reciprocal polynomials of odd degree*, J. Inequal. Pure Appl. Math. **4** (2003), no. 3, Article 60, 8 pp.

[11] ———, *Self-inversive polynomials whose zeros are on the unit circle*, Publ. Math. Debrecen **65** (2004), no. 3-4, 409–420.

[12] M. Marden, *Geometry of Polynomials*, Second edition. Mathematical Surveys, No. 3 American Mathematical Society, Providence, R.I. 1966

[13] I. D. Mercer, *Unimodular roots of special Littlewood polynomials*, Canad. Math. Bull. **49** (2006), no. 3, 438–447.

[14] K. Mukunda, *Littlewood Pisot numbers*, J. Number Theory **117** (2006), no. 1, 106–121.

[15] R. Salem, *Algebraic Numbers and Fourier Analysis*, D. C. Heath and Co., Boston, Mass. 1963

[16] A. Schinzel, *Self-inversive polynomials with all zeros on the unit circle*, Ramanujan J. **9** (2005), no. 1-2, 19–23.

[17] C. L. Siegel, *Algebraic integers whose conjugates lie in the unit circle*, Duke Math. J. **11** (1944), 597–602.

[18] C. J. Smyth, *On the product of the conjugates outside the unit circle of an algebraic integer*, Bull. Lond. Math. Soc. **3** (1971), 169–175.

[19] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall Series in Automatic Computation Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964.

Department of Mathematics and Informatics
Vilnius University
Naugarduko 24, Vilnius LT-03225, Lithuania
E-mail address: pdrungilas@gmail.com