Another proof that there are infinitely many primes

There are many proofs that there are infinitely many primes. An argument by Chaitin [1] shows, roughly speaking, that if there were only finitely many primes, then there would not be enough prime factorizations to represent large integers. In this note, we give a short explicit argument based on this idea. We show that if there were only k primes, then the integers from 1 to k^{3k} would have fewer than k^{3k} different prime factorizations.

Suppose there are only k primes

$$p_1 = 2, \qquad p_2 = 3, \qquad p_3 = 5, \qquad \ldots, \qquad p_k.$$

Since $2, 3, 5, 7$ and 11 are prime, we have $k > 4$. We will use "lg" to denote the base 2 logarithm. Note that we have the crude bounds $k > \lg k$ and $k \lg k > 1$.

Let $N = k^{3k}$. Given any positive integer $n \leq N$, we can write

$$n = 2^{a_1} 3^{a_2} 5^{a_3} \cdots p_k^{a_k}$$

and for each j, we have

$$2^{a_j} \leq p_j^{a_j} \leq n \leq N,$$

implying $a_j \leq \lg N$. So a_j is between 0 and $\lg N$, so there are at most $1 + \lg N$ possibilities for each a_j. We then observe

$$1 + \lg N = 1 + 3k \lg k < 4k \lg k < k \cdot k \cdot k = k^3,$$

so there are strictly fewer than k^3 possibilities for each a_j, and hence fewer than $(k^3)^k$ possibilities for the tuple $(a_1, \ldots, a_k)$. That is, there are fewer than k^{3k} possibilities for the prime factorization of n, so it is not possible to construct prime factorizations for all positive integers $n \leq k^{3k}$.

References

[1] G.J. Chaitin, *Toward a mathematical definition of life*, in *The Maximum Entropy Formalism*, R.D. Levine and M. Tribus, eds., MIT Press, Cambridge, 1979, pp. 477-–498.